Remaining at Yield During Unloading and Other Unconventional Elastic-Plastic Response[1]

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A rather special time-independent or elastic-plastic response is proposed in which, although there is elastic response to unloading, the material remains at yield for all or a significant portion of the unloading path following plastic deformation. In the most elementary form, the material exhibits no memory of prior plastic deformation; the current state of the material is given solely *by the current state of stress. A simple but unconventional field of plastic moduli then can be chosen to produce a limit surface that cuts through a nested set of yield surfaces and to model critical aspects of the behavior of sand.*

A Material Remaining At Yield During Unloading

At each state of stress, suppose, as in conventional plasticity theory, there is a local surface passing through the stress point which, for any small stress increment vector, separates the domain of purely elastic response from the domain of plastic (actually elastic-plastic) response (Fig. 1(*a*)). These local surfaces are further supposed to join to form a set of continuous surfaces in stress space. A nested set of these surfaces (Fig. 1(*b*)) then resembles a sequence of successive yield surfaces or loading surfaces for a stress-hardening plasticity theory, analogous to but very different from the Mises or J_2 isotropic hardening idealization. As in the conventional formulation, when the stress point moves about on one of these surfaces in stress space (neutral loading), purely elastic deformation takes place. Any motion of the stress point directed toward the outside will produce plastic as well as elastic deformation. A move directed inward gives purely elastic response.

However, a move inward has very different consequences in conventional plasticity and in the special formulation proposed here. Conventional plasticity does appropriately idealize the macroscopic behavior of metals and alloys at temperatures that are a small fraction of their melting point. The current yield or loading surface remains unchanged as the stress point moves inside it. Plastic deformation will not take place again until the stress point reaches and then penetrates the yield surface established by the prior history of loading (Fig. 2(*a*)). In the proposed formulation, there may or may not

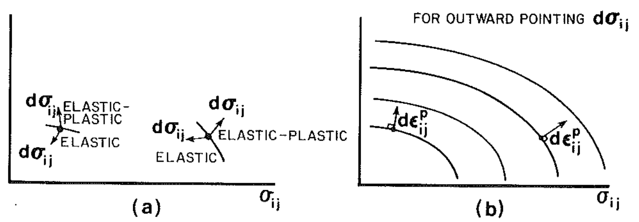

Fig. 1 The current yield surface passes through the current stress point and locally separates the domain of purely elastic response from the domain of elastic-plastic response

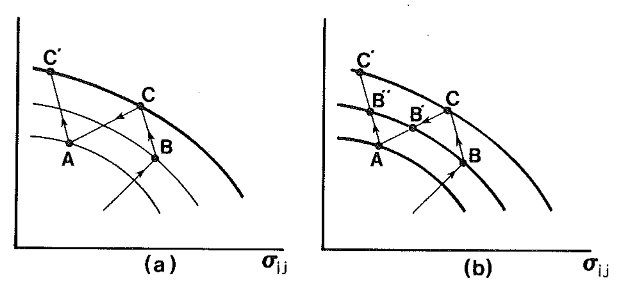

Fig. 2 In conventional plasticity (2a) path CAC′ is purely elastic, in the proposed formulation (2b) path CB′A is elastic but AB″C′ is elastic-plastic

be a purely elastic domain of stress. Outside of that domain, if one is postulated, or throughout the entire stress space if one is not, the current yield surface always passes through the current stress point, whether the motion of the stress point represents loading or unloading (Fig. 2(*b*)). Inelastic deformation occurs immediately when the stress point once again moves outward.

For the simplest form of no purely elastic domain and no history dependence or memory of prior plastic deformation, there is no change in either the family of yield surfaces or the

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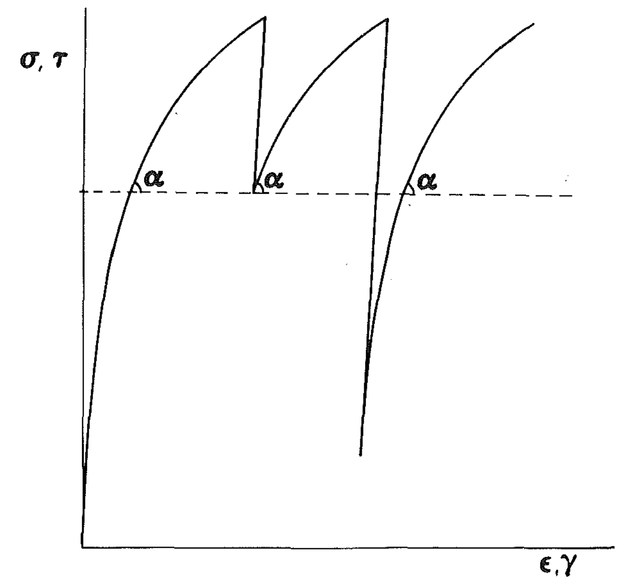

Fig. 3 Successive stress-strain curves for uniaxial stress or shear are the initial curve translated along the strain axis in the simplest model

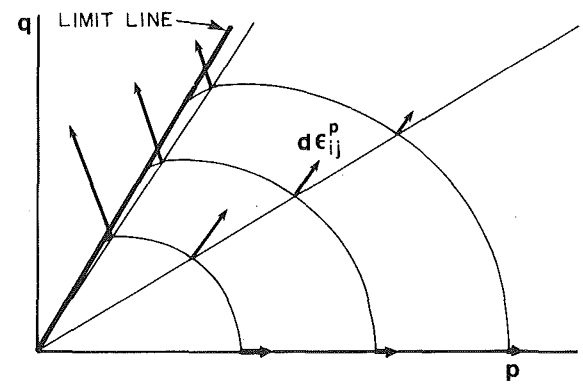

Fig. 4 Pictorial representation for sand of the nested set of yield surfaces, the limit line, and the field of plastic moduli (shown by the $d\epsilon^p_{ij}$ associated with a constant value of $n_{pq}d\sigma_{pq}$)

field of plastic moduli. In one dimension (simple tension or simple shear), as shown in Fig. 3, the slope of the stress-strain curve for reloading at each stress level is the same as for the initial loading. Each reloading curve is simply the original loading curve translated along the strain axis.

The representation resembles in part the proposal by Drucker and Palgen (1981) for the stable cyclic behavior of metals, without either kinematic hardening or isotropic workhardening or softening (see their Fig. 3). Other earlier proposals, such as those by Eisenberg and Phillips (1971), Dafalias and Popov (1975), and Hashiguchi (1980) also have some of the features of this one with plastic deformation occurring inside of or on surfaces labeled as bounding surfaces or loading surfaces, respectively, both of which are distinguished from yield surfaces enclosing an elastic domain. These earlier idealizations were developed to simulate the behavior termed *metarecovery* by Cherian, Pietrokowsky, and Dorn (1949) in their study of commercially pure aluminum. For uniaxial stress, the reloading curve is asymptotic to what would have been the uninterrupted virgin loading curve. The simple form of the idealization proposed here is a special case of what they termed orthorecovery; the reloading curve is parallel to the original curve (Fig. 3).

As illustrated, each yield surface in stress space will be taken to be convex and the material to be stable in the small in the forward sense (positive elastic and positive plastic moduli). Yet both an instability in the small for a cycle of unloading followed by reloading, and an instability in the large, are present in the sense of Drucker (1951). Work can be extracted from the material and the system of forces or stresses acting upon it in a cycle of stress, or equally well in a cycle of strain, a characteristic shared with metarecovery idealizations. Consequently, neither the comforting equivalent of the shakedown theorem nor the lower bound theorem of perfect plasticity holds. Inelastic deformation will continue to grow indefinitely with constant increments as the stress level is oscillated between fixed limits. Such continual deformation is associated with a simple frictional system, as a block subjected to fluctuating tangential and normal forces that cause it to be alternatively at and just above incipient sliding on a rough surface, or a fully or partially continuously annealing metal or alloy, but not with a metallic system well below its annealing temperature.

The possible value of so overly simple and yet unconventional an idealization for metals or for granular media remains to be demonstrated. The reader is urged to both withhold judgment until a few illustrative examples are explored and to expect no greater generality of response than from any simple idealization of inelastic behavior.

The Stress-Strain Relation

The behavior of the material at each state of stress is supposed to be stable in the small in the forward sense for an outward motion of the stress point and for a cycle of elastic return. Then, at a smooth point on the yield surface, the plastic strain increment vector $d\epsilon^p_{ij}$ produced by the outward stress increment vector $d\sigma_{ij}$ is normal to the current yield surface, F = constant:

$$d\epsilon^p_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{1}$$

or

$$d\epsilon^p_{ij} = G \frac{\partial F}{\partial \sigma_{ij}} \left(\frac{\partial F}{\partial \sigma_{pq}} \, d\sigma_{pq} \right) \tag{2}$$

where in the simplest formulation both G and F are functions of stress only.

In a more transparent form for loading ($n_{pq}d\sigma_{pq} > 0$),

$$d\epsilon^p_{ij} = \frac{1}{K_p} \, n_{ij} \, (n_{pq} d\sigma_{pq}) \tag{3}$$

where n_{ij} is the unit normal $\partial F/\partial\sigma_{ij} \,/\, \sqrt{[(\partial F/\partial\sigma_{rs}) \; (\partial F/\partial\sigma_{rs})]}$, $n_{pq}d\sigma_{pq}$ is the normal component of the stress increment vector $d\sigma_{pq}$, and the plastic modulus K_p is a scalar function of the current stress only.

$$\frac{1}{K_p} = \left(\frac{\partial F}{\partial \sigma_{rs}} \; \frac{\partial F}{\partial \sigma_{rs}} \right) G \tag{4}$$

The field of plastic moduli, K_p, remains fixed throughout stress space in this simplest version of the proposed behavior. More elaborate formulations, which take one or more aspects of the history of deformation into account, are easily devised. It seems premature at this stage to go much beyond the exploration of the possible value of the simple version.

Modeling the Behavior of Sand

Consider a material subjected to shear stress q and hydrostatic pressure p having a set of nested yield surfaces in $q-p$ space as shown in Fig. 4. The surfaces are chosen so that the normal to each is constant in direction along a radial line from the origin. The modulus K_p is taken to increase with

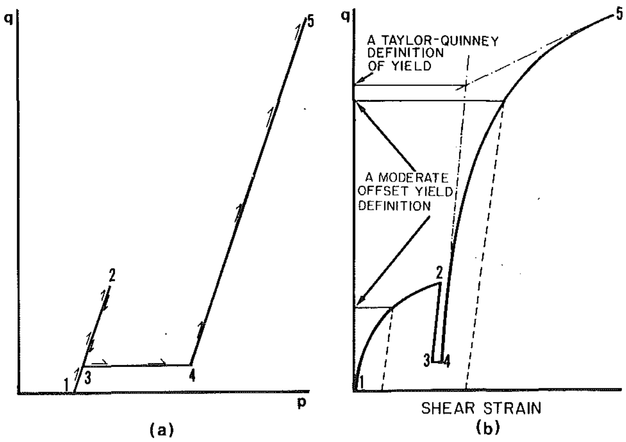

Fig. 5 (*a*) Type "A" loading-unloading-reloading stress path and (*b*) corresponding stress-strain curve

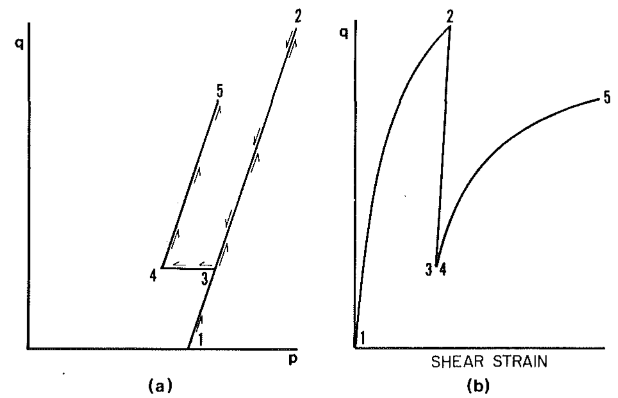

Fig. 6 (*a*) Type "B" loading-unloading-reloading stress path and (*b*) corresponding stress-strain curve

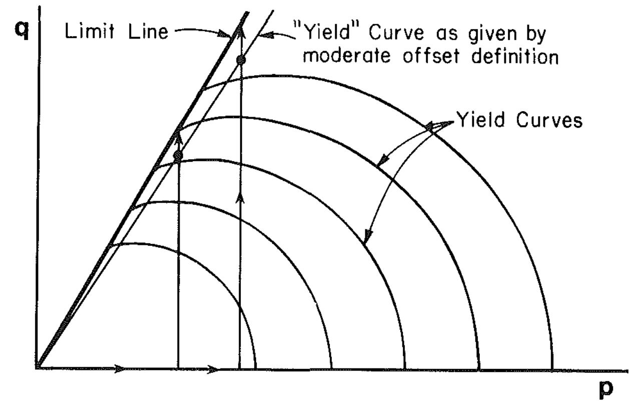

Fig. 7 The "yield" surface as defined in Fig. 5(*b*) by a moderate offset definition cuts across the nested set of yield surfaces that generates Fig. 5(*b*)

radial distance from the origin in accord with the stiffening of the response of sand with increase in hydrostatic pressure. The modulus is taken to decrease with increase in the angle of inclination of the radius to hydrostatic pressure axis, slowly at first and more and more rapidly later. K_p goes to zero at a limiting line, which is beyond the zero dilation line, in accord with the failure condition for sand. This inelastic response is indicated schematically in Fig. 4 by the direction and relative length of the plastic strain-increment vectors produced by the same normal increment of stress $n_{pq}d\sigma_{pq}$ at each and every point. It is worth noting that because F and G are purely stress dependent, satisfaction of the consistency condition is automatically assured in contrast to the more difficult approach commonly adopted for isotropic hardening soil models (see Mroz, 1984, for a survey of current approaches).

The proposed choices, motivated by the observed behavior of sand, are instructive because they are so very different in so many ways from the familiar behavior of metals and their customary idealizations as elastic-plastic. Note that the limiting line from the origin, taken as straight and radial for simplicity, intersects the successive yield surfaces at an appreciable angle and is not a yield surface itself. However, if the plastic moduli are comparable to the elastic until the limiting line is approached, and if a large strain offset definition of yield is employed, instead of the infinitesimal plastic strain definition, the initial "yield" surface so defined would be rather close to the limit line. More generally, for the model of sand as described, initial and subsequent "yield" surfaces determined by a moderate plastic strain offset or a Taylor-Quinney (1931) definition so suitable for metals, have entirely different shapes from the nested set of yield surfaces that generate them. Figures 5 and 6 each show a loading-unloading-reloading path and the corresponding stress-strain curves that the representation gives. Figure 7 indicates schematically the shape of the surface that would be defined as a "yield" surface with either a moderate (but constant) offset definition or with a Taylor-Quinney definition.

Comments on the Correspondence to Reality

Too much should not be expected from any simple model, especially one that is unconventional and so by definition will not give the traditional results that have served so well for so long for the plastic deformation of metals and alloys. Certainly this representation, unlike conventional plasticity, is not intended to capture the essence of the response of most structural metals to a generally outward path of loading in stress space followed by a full or partial unloading and then reloading. However, it is worth noting that the behavior of cyclically stabilized metals and alloys between two fixed limits of strain can be given reasonably well by such a simple formulation. Ratcheting will be predicted for oscillation between two unsymmetric stress limits of opposite sign: too much ratcheting for a large number of cycles, but not so bad an approximation for just a few cycles.

It is by no means clear that such a representation is adequate for sand, despite the qualitative agreement of the stress-strain curves of Figs. 5 and 6 with experimental results. However, the concept of the yield surface following the stress point for unloading as well as loading has some physical appeal. The response of a granular medium to shear at a given void ratio does depend upon the hydrostatic pressure. Furthermore, the void ratio does not change appreciably with inelastic shear deformation until the deformation becomes quite large. At any given void ratio, a reduction in hydrostatic pressure surely will lower the shear stress level required for any given ratio of incremental inelastic shear to volumetric strain; an increase will raise the shear stress required. If it makes sense to think of *yield surfaces in stress space for sand (and that's a very big if)*, then it is not unreasonable to associate a fixed ratio of shear stress to hydrostatic pressure with a corresponding ratio of inelastic strain increments as sketched in Fig. 4. It is also not unreasonable for the current yield surface to go through the current stress point for an unloading as well as a loading path. Inelastic deformation would be observed at a point O in Fig. 8 for each of the outward paths drawn through that stress point, even though the operative segment DO of path CDO is well inside the largest yield surface reached earlier and so its extension beyond O would be thought of as elastic in conventional plasticity.

In a sequence of careful experiments by Poorooshasb,

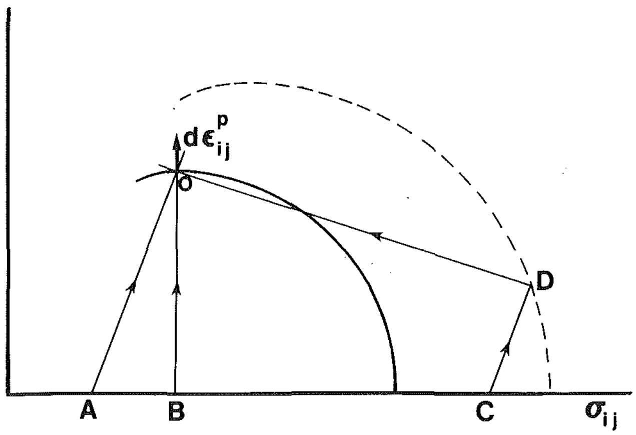

Fig. 8 Plastic deformation at O for the extension of path CDO as well as of AO and BO

Holubec, and Sherbourne (1966, 1967), the direction of the plastic strain increment vector at O was observed to be independent of the direction of the path of loading through O. It was, in fact, these experimental observations and those of Tatsuoka and Ishihara (1974a, 1974b) which suggested that the combination of the yield curve following the stress point on unloading, a fixed field of plastic moduli, and nested surfaces might provide a suitable first approximation for sand. As stated by Tatsuoka and Ishihara (1974a), "the sample behaves as if it were in a virgin state when the mobilization of shear stress is reversed in direction".

Modifications of the Proposal to Exhibit More Complex Behavior

As mentioned earlier, a domain of purely elastic response can be postulated that is fixed in stress space or that expands or contracts in accord with the inelastic history of the material. In the most convenient but not necessarily appropriate form, the elastic domain will be bounded by one of the nested yield surfaces. If desired, the elastic domain can be taken to expand as the stress paths repeatedly penetrate successive yield surfaces. Should hardening of the material occur within any surface, such as the farthest out yield surface reached in prior loading, the plastic modulus K_p at interior points can be taken to increase. The increase may be chosen to depend upon the distance from the interior stress point to a conjugate point on that surface, or the number of times the interior yield surfaces have been crossed in an outward direction, or the extent of the prior plastic deformation, or the decrease in void ratio, or any other measures believed important separately or in combination. Similarly, if softening of the material takes place as a result of significant dilation or the accumulation of plastic distortion, the plastic modulus can be decreased appropriately.

Another complexity that can be added if needed to improve agreement with experiment is the alteration of the limiting line in $q-p$ space, or the limiting surface in a more general stress space to follow any large change in void ratio or fabric. Also, the limiting line need not be taken as straight or to pass through the origin.

Concluding Remarks

Two entirely independent special modes of elastic-plastic response have been described within the framework of stability in the small in the forward sense. The first is a rather radical departure from the conventional formulation of plasticity theory. In the proposed formulation, the current yield surface continues to pass through the current stress point, whether the motion of the stress point represents loading or unloading. Inelastic deformation occurs immediately when the stress point once again begins to move outward. Experimental results on sand provided the motivation for this basic assumption which leads to instability in the large for some important paths of loading as well as instability in the small for an unloading-reloading cycle. One of the unpleasant aspects of this instability is that the response to imposed loads of a body under an inhomogeneous state of stress can be highly dependent upon the initial state of stress, even when the subsequent inelastic strain becomes quite large. However, granular and frictional media do exhibit such instability in the large and corresponding sensitivity to initial stress. Although metallic systems rarely do, the formulation is of some use for metals that self anneal or have been stabilized under cyclic loading.

The second major departure from conventional plasticity theory does in principle lie within the scope of the general theory for fully stable workhardening materials. A limit line or surface is produced by choosing the plastic modulus as approaching zero as the stress point moves outward toward the limit surface. However, in contrast with the traditional approach, the limit surface chosen here cuts across and is not one of the family of initial or subsequent yield or loading surfaces. For a workhardening metal or alloy, such a choice of a field of plastic moduli also would exhibit the same selected limit surface for generally outward loading. It would not, however, for loading appreciably in any one direction that does not intersect the limit surface in stress space, then unloading to a state of pure hydrostatic pressure and reloading in another direction that does. The subsequent outward loading path could reach and cross the limit surface as a purely elastic path and the supposed limiting surface would not limit.

It is the combination, a yield surface following the stress point for unloading as well as loading and a limiting surface in stress space established by the approach to zero of the plastic modulus on a very different surface from the set of yield surfaces, that provides the promise of a useful consistent model for the response of sand and other granular media. The limit surface then is a surface in stress space that will not be passed by the stress point on reloading following loading and unloading to a state of pure hydrostatic stress. All is not quite as it should be in this simple formulation because the limit surface can be penetrated by a neutral or a somewhat unusual inward loading path starting near the limit surface. Further development is needed to circumvent this difficulty and to include other significant aspects of material behavior as needed.

The final point to emphasize is that definitions of yield are arbitrary and can be misleading. The calculated curves, Figs. 5 and 6, for elastic-plastic response of the simple model, with its fixed nested set of yield surfaces in stress space, Fig. 4, match the results of the usual triaxial tests on sand very well. Consequently, they do give the customarily reported set of moderate or large offset yield surfaces, Fig. 7, which bear no resemblance at all to the (infinitesimal offset) yield surfaces that generated these quite differently defined "yield" surfaces. The conclusions an observer is likely to draw from the calculated data, treated as though they were experimental, is that the inelastic stress-strain relation does not obey normality to the current yield surface, i.e., it is nonassociated. Of course, the fact that an associated flow rule gives data that could reasonably be interpreted as nonassociated does not demonstrate that nonassociated flow rules are simply artifacts of interpretation. The associated flow rule results do, however, provide some food for thought as well as a stabilizing option for computation.

Acknowledgments

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